A PROBABILISTIC MODEL FOR DETERMINING AN OPTIMUM POLARIS TENDER LOAD LIST

RICHARD J. DAMICO
and
NEIL L. HARVEY

A PROBABILISTIC MODEL FOR
DETERMINING AN OPTIMUM POLARIS TENDER LOAD LIST

* * * * *

Richard J. Damico

and

Neil L. Harvey

A PROBABILISTIC MODEL FOR DETERMINING AN OPTIMUM POLARIS TENDER LOAD LIST

by

Richard J. Damico

Lieutenant, United States Navy

and

Neil L. Harvey

Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

BACHELOR OF SCIENCE
with major in
OPERATIONS RESEARCH

United States Naval Postgraduate School
Monterey, California

1 9 6 3

A PROBABILISTIC MODEL FOR DETERMINING AN OPTIMUM POLARIS TENDER LOAD LIST

by

Richard J. Damico

and

Neil L. Harvey

This work is accepted as fulfilling

the thesis requirements for the degree of

BACHELOR OF SCIENCE

with major in

OPERATIONS RESEARCH

from the

United States Naval Postgraduate School

ABSTRACT

A mathematical model is developed for determining an optimum submarine tender load list for the POLARIS (FBM) weapon system. Military essentiality of spare parts and the stowage capacity of the tender play prominent roles in the development of the model. The measure of effectiveness which is maximized is the average number of demands filled over a given period of time. This is equivalent to minimizing the number of stockouts of an item. An example of the use of the model is included for the case where the demand for items is characterized by the normal distribution. The paper includes a computer flow chart and program to aid in implementing the model.

The authors wish to express their appreciation for the assistance and encouragement given them by Associate Professor Julius H. Gandelman of the U. S. Naval Postgraduate School. The authors are also indebted to Miss Patricia Hoang for her assistance in programming the model on the CDC 1604 computer.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

GLOSSARY OF TERMS

i	=	a given item where $i = 1,2,3.......N$ (the entire list of spare parts is drawn up and then a number is assigned to each spare part to identify it).
z_i	=	amount of item i to be stocked.
x_i	=	demand for the i^{th} item.
$f_i(x_i)$	=	probability that an amount x_i will be demanded during the period under consideration.
$E[x_i]$	=	expected demand for item i ; $= m_i$.
$b(z_i)$	=	expected shortage of item i.
$s(z_i)$	=	average number of i^{th} item supplied to meet demand.
λ	=	Lagragian multiplier.
c_i	=	cube (volume) of the i^{th} item.
C	=	total cube available for spare parts stowage on tender.
C'	=	total cube associated with any λ for which $\sum_i c_i z_i \neq C$.
M_i	=	essentiality (military worth) number assigned to i^{th} item.
σ_i	=	deviation of i^{th} item from the mean.
m_i	=	mean of the i^{th} item.
P	=	characteristic number associated with the i^{th} item.

1. Introduction.

The trend toward the development and utilization of highly complex, sophisticated weapon systems within the Navy has generated an ever-increasing requirement for more efficient logistical support. This paper is an attempt to develop a mathematical model that can be used as a decision aid for one aspect of the logistics problem associated with the POLARIS system. The model provides a method for determining the items and associated quantities that should be stocked aboard the supporting tender.

The requirement for more effective tender load lists is even more evident when viewed in conjunction with the POLARIS mobile base concept.

In the event of enemy attack, the Navy's immediate POLARIS retaliatory capability will have been dependent on the ability of the supporting tender to provide the necessary SSB(N) replenishment and repair requirements. This aspect of the tender function is further complicated by the limitations of stowage space available for stocking the required spare parts. As a result of this space limitation, tender load lists must reflect SSB(N) requirements, and tender re-supply procedures must be closely coordinated and evaluated to insure that SSB(N) requirements are being met in an effective and efficient manner.

The concept of military worth is incorporated in the model to reflect the assignment of priorities between the various items. These essentiality numbers are an attempt to define the military worth of a particular item with reference to the performance of the integrated weapon system. Essentiality factors will be assigned arbitrarily; however, it should be apparent that actual MEC numbers could have been utilized for illustrative purposes.

The measure of effectiveness to be maximized will be the average number of demands for spare parts that are filled by the tender over some period, as a function of the military essentiality of the various items. This is equivalent to minimizing the number of stockouts of an item.

It is obvious that if the demand for an item was known exactly, the problem would be simplified considerably subject to the space limitation of the tender. Although demand cannot be predicted with certainty it may be possible to estimate the probability distribution of demand for each item on the basis of past usage data. In the event that usage data are not available, the model can be utilized by assuming a probability distribution of demand for the item(s), based upon experience and judgment. This would be the situation encountered in developing an initial tender load list for a new class of submarines. As usage data become available the tender load list can be continually updated to reflect actual demand, utilizing automatic data processing equipment. Appendix III contains a computer flow diagram and program that will accomplish this objective.

Although this paper is primarily concerned with presenting a method for determining an optimal tender load list, certain aspects of the tender re-supply problem are included for consideration.

2. Assumptions

The following simplifying assumptions are introduced in order to illustrate the proposed method.

1) The model is concerned with spare parts and does not consider dry stores and other consumable items. As a result, C refers to the cube available for spare parts stowage.

2) x_i is a continuous variable, and its associated probability density function $f_i(x_i)$ possesses the properties required for the existence of derivatives and integrals.

3) The demand for each item is independent of the demand for every other item.

4) The probability distribution of demand is known for each item over the time period considered.

5) The list of items considered reflects only those for which demand has been generated. If demand for an item stops, then the item is removed from stock and a new load list is generated. Conversely, if demand for a new item is generated, it is added to the load list and a revised $Z = \left| z_1, z_2, .. z_N \right|$ is determined by means of the model.

6) The development of the model is independent of the re-supply phase of the tender operation. The load list determined is a function of demand over some period during which the tender must provide support without being re-supplied. Theoretically the tender

is completely depleted of spare parts at the end of the period considered. The effect of re-supply as a function of the desired supply objective will be considered in a later section.

7) The utility values defined in Appendix I are additive.

3. Presentation of the Model.

This section presents the essential concepts of the proposed model. Appendix I contains a detailed mathematical formulation of the results of this section.

The expectation, $E[\ ]$, of x_i with respect to the probability function $P[\cdot]$ gives the following expression for the average amount of item i demanded:

$$(1)\quad E[x_i] = \int_0^{\infty} x_i \; f_i(x_i)\, dx_i ,$$

where $x_i =$ demand for the i^{th} item.

(The limits of integration range from 0 to ∞ since only items for which a demand is generated are considered)

Consider the following situations:

(a) $x_i < z_i$,

(b) $x_i > z_i$,

where $z_i =$ amount of item i to be stocked.

If demand for an item is less than the amount of the item that isstocked then the tender can meet demand. However, if demand exceeds the stock level then only an amount z_i can be supplied. If this condition occurs $(x_i - z_i)$ represents the shortage (backorder) associated with that item. The expected shortage of an item can be expressed as a function of the amount of the item stocked.

$$(2)\quad b_i[z_i] = \int_{z_i}^{\infty} (x_i - z_i)\; f_i(x_i)\, dx_i \qquad \text{For } x_i > z_i ,$$

$$= 0 \qquad\qquad \text{otherwise} .$$

The expected value of the amount actually supplied to meet demand is given as a function of the amount stocked on the tender. This results in the following expression:

$$(3) \quad s_i[z_i] = \int_0^{z_i} x_i f_i(x_i)\, dx_i + z_i \int_{z_i}^{\infty} f_i(x_i) dx_i$$

Manipulation of equations (1-3) confirms the intuitive concept that the expected demand must equal the expected demands filled plus the expected shortages.[1]

$$(4) \quad E[x_i] = s_i[z_i] + b_i[z_i]$$

As mentioned previously, the measure of effectiveness is a function of the military essentiality of the item as well as the demands filled by the tender. This requirement is met by defining $M(z_1, z_2, \ldots z_n) = \sum_{i=1}^{N} M_i \, s_i(z_i)$ in accordance with assumption (7). The problem is now one of determining the set $Z = \| z_1, z_2, \ldots z_N \|$ such that

$$M(z_1, z_2 \ldots z_N) = \sum_{i=1}^{N} M_i \, s_i(z_i)$$

is maximized subject to the constraint $\sum_{i=1}^{N} c_i z_i = C$, $z_i > 0$.

Equation (5)(below) is derived in Appendix I and is a general result of the considerations mentioned in this section. This equation provides the means for determining an optimum tender load list.

$$(5) \quad \int_0^{z_i} f_i(x_i)\, dx_i = 1 - \lambda\left(\frac{c_i}{M_i}\right), \qquad i = 1, 2 \ldots . N,$$

where λ is a lagrangian multiplier with values in the range $\min\left(\frac{M_i}{c_i}\right) \geq \lambda \geq 0$

subject to:

$$\sum_{i=1}^{N} c_i z_i = C, \quad z_i > 0.$$

---

[1] Appendix I

4. General procedure for utilizing the model.

The general procedure for solving equation (5) is to assume a value for λ corresponding to min $\left(\frac{M_i}{C_i}\right)$, and compute the set Z by using the appropriate cumulative distribution function table. The C' obtained (where C' results from using a λ other than that for which $\sum^{N} c_i z_i = C$) when the z_i are substituted into the constraint equation ($\sum_{i=1}^{N} c_i z_i = C'$) determines the minimum required stowage space aboard the tender as a function of demand over the demand period associated with $f_i(x_i)$. (As λ increases, C' decreases. Since min $\left(\frac{M_i}{C_i}\right)$ is the largest permissible value of λ, the corresponding C' is a minimum.) It should be noted that for any λ in the range min $\left(\frac{M_i}{C_i}\right) \geq \lambda \geq 0$, every $z_i > 0$. This automatically sets a minimum stowage requirement on the tender of at least the sum of the cube of each item.

Under the assumption that C is known for the tender, a comparison is made between this value and C'. If $C > C'$ an iterative solution is required to determine the λ which results in $C' = C$. The corresponding z_i's obtained are the optimum load list.

If, on the other hand, $C < C'$ the model indicates that the tender cannot effectively meet demands without re-supply during the period under consideration. In the event this occurs several decision criteria can be adopted. If demand for each item is relatively uniform, one solution would consist of decreasing the tender load list time requirement.[1] For example: Let $C' = 15{,}000$ cu.ft. represent a ninety-day load list minimum requirement. If $C = 5{,}000$ cu. ft., then the tender can in effect maintain a 30 day load list by decreasing each z_i proportionally.

This criterion has the advantage of maintaining a load list which consists of all items that are demanded. The most obvious disadvantage is the increased re-supply requirement.

Another possibility is the elimination of items from the load list. This requires a decision criterion to select the item(s) which should be deleted. Since C' decreases as λ increases, the model indicates that the item corresponding to $\min(\frac{M_i}{c_i})$ should be the first item removed from stock. If $(C' - c_i a_i = C \; ; \; a_i < z_i)$ then an amount $(z_i - a_i)$ can be maintained for this item. If $(C' - c_i z_i > C)$ then the item corresponding to $\min(\frac{M_i}{c_i})$ is eliminated completely. A new $\min(\frac{M_i}{c_i})$ is selected from the remaining list of N-1 items and the process is repeated. When a λ can be found such that $C = C'$ the z_i's have been determined for this decision criterion. This particular policy insures that the items with the highest essentiality to cube ratio $(\frac{M_i}{c_i})$ are retained on board.[2]

---

[1] That period during which the tender must be able to operate without benefit of re-supply and still maintain effective support capability.

[2] Note that other policies could have been chosen. For example, one might choose a policy that insures that items with high values of P are retained on board, while items with low values of P are eliminated.

It permits tender support to be maintained at some reduced effectiveness level. (This reduced level can be a significant one). In effect, this is analogous to an unexpected demand situation and will be considered further in connection with tender re-supply requirements.

The iterative nature of the solution and the number of items comprising load lists indicates the necessity for utilizing electronic computers if the system is to be incorporated effectively. Appendix III contains a computer description and flow chart for the model.

5. Example and General Results:

A numerical example will serve to illustrate the procedure for determining an optimal tender load list utilizing the proposed model. Consider the case where the demand for each item is characterized by the normal distribution. From equation (5)

$$\frac{1}{\sqrt{2\pi}} \int_0^{\frac{z_i - m_i}{\sigma_i}} e^{-\frac{1}{2}y^2} dy = \Phi\left(\frac{z_i - m_i}{\sigma_i}\right) = 1 - \lambda\left(\frac{c_i}{M_i}\right),$$

subject to the available stowage constraint. It should be noted that each item could have different demand distributions and the model would still be valid. The only requirement for utilizing the model is that the density functions be continuous.[1]

Table (1) contains the parameter values pertinent to the example. In accordance with the procedures of the previous section, an available tender capacity of 10,152 is assumed. It should be apparent that due to the iterative nature of the solution, $\sum_{i=1}^{N} c_i z_i$ will be approximately equal to 10,152 for an optimum set Z. (The most obvious reason is the need for rounding off the values obtained to intergers. In order to insure that the space constraint is met, the z_i's obtained should be rounded off to the lower interger with the result that $\sum_{i=1}^{N} c_i z_i \approx C$).

The values assigned to the items listed in Table (1) reflect varying combinations of essentiality, cube, expected demand and

[1]Another model could be developed which would utilize discrete functions.

standard deviation.

The essentiality valves in Table 1 were arbitrarily chosen to show that items 1-8 are approximately four times as essential as items 9-16. In a real life problem involving a large number of items, a scheme using the military essentiality codes could be adapted for use in this model.

Table 1

ITEM	$E[x_i]$	σ_i	M_i	c_i	$\frac{M_i}{c_i}$	$\frac{c_i}{M_i}$	$P = \frac{m_i}{\sigma_i} \cdot \frac{M_i}{c_i}$
1	100	12	9	12	.75	1.33..	6.25
2	100	3	9	12	.75	1.33..	25
3	50	12	9	12	.75	1.33..	3.12
4	50	3	9	12	.75	1.33..	12.5
5	100	12	9	3	3	.333..	25
6	100	3	9	3	3	.333..	100
7	50	12	9	3	3	.333..	12.5
8	50	3	9	3	3	.333..	50
9	100	12	2	12	.1666..	6	1.38
10	100	3	2	12	.1666..	6	5.55
11	50	12	2	12	.1666..	6	0.69
12	50	3	2	12	.1666..	6	2.76
13	100	12	2	3	.666..	1.5	5.55
14	100	3	2	3	.666..	1.5	20
15	50	12	2	3	.666..	1.5	2.76
16	50	3	2	3	.666..	1.5	11.1

In order to determine the value of M once an optimum set Z is obtained, equation (3) must be utilized. For the case when demand is characterized by the normal distribution,

$$(3) \quad S_i[z_i] = \int_0^{z_i} x_i f_i(x_i)\,dx_i + z_i \int_{z_i}^{\infty} f_i(x_i)\,dx_i =$$

$$(3') \quad m_i\left[1-\lambda\left(\frac{c_i}{M_i}\right)\right]+z_i\left[\lambda\frac{c_i}{M_i}\right]+\frac{\sigma_i}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{m_i}{\sigma_i}\right)^2}-\frac{\sigma_i}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z_i-m_i}{\sigma_i}\right)^2}$$

Appendix II contains a detailed derivation of equation (3'.).

Since equation (5) requires $z_i > 0$ this implies that for the case of the normal distribution equation (5) and (3') are only valid when $\frac{m_i}{\sigma_i} > 3.87$.[2] (Fig. 1)

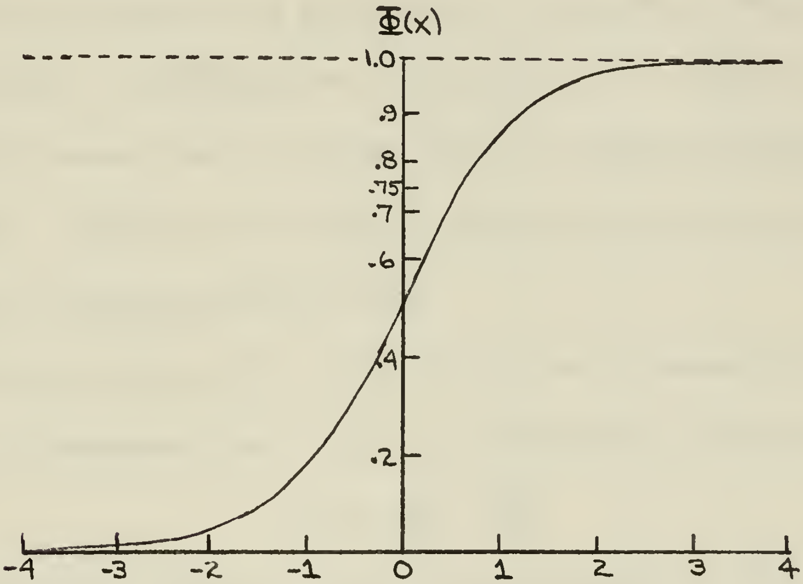


Fig. 1: Graph of the normal distribution function $\Phi(x)$

Since $f_i(x_i) > 0$, $\Phi\left(\frac{z_i-m_i}{\sigma_i}\right) > 0 \implies \frac{z_i-m_i}{\sigma_i} > -3.87$.

$\therefore$ For $z_i > 0$, the ratio $\frac{m_i}{\sigma_i} > 3.87$ must hold. Examination of Table 1 indicates that this condition has been satisfied.

Due to the requirement above, the third term of equation (3')

---

[2] The value for $\frac{m_i}{\sigma_i}$ is dependent on the accuracy of the tables used. The value of 3.87 was obtained from the Chemical Rubber Company Standard Mathematical Tables.

can be ignored in this example since its contribution is insignificant for the values assumed.

Table 2 is a presentation of the results obtained when the parameters shown in Table 1 are used in the model. The optimal set $Z_0 = \| z_1, z_2, \ldots z_{16} \|$ which most nearly satisfies the arbitrary constraint of 10,152 cu. ft. tender capacity was found by using $\lambda = .0416$.

Note that items 9-12 generally lag in percent of expected average demand filled $S_i(z_i)/E[x_i]$, for each value of λ. This is intuitively appealing because these are items with low military essentiality and high cube. The items with high essentiality and low cube (5-8) are stocked at consistently high levels.

Figures 2a thru 2d are plots of percent of expected average number of demands filled, versus a characteristic number P, defined as

$$P_i = \frac{m_i}{\sigma_i} \cdot \frac{M_i}{c_i}$$

Since each item is characterized by the parameters m, σ, M and c, each item is also characterized by P. Note that for different values of C' the curve shifts somewhat but retains the same general shape. In each case as the characteristic number increases, the percent of expected average number of demands filled increases.

The effect of variability of demand for this example is investigated in Figure 3 by observing the behavior of items 9-12. Since items 9 and 10 (also 11 and 12) differ only in their values of we would expect any differences in their optimal stock levels (z_i)

Table 2

	$\lambda = .1666$					$\lambda = .1250$				
	$\Sigma c_i z_i = 8088$			$\Sigma M_i s(z_i) = 6317$		$\Sigma c_i z_i = 9341$			$\Sigma M_i s(z_i) = 6516$	
i	z_i	$s(z_i)$	$s(z_i)/E[X]$	$c_i z_i$	$M_i s(z_i)$	z_i	$s(z_i)$	$s(z_i)/E[X]$	$c_i z_i$	$M_i s(z_i)$
1	109.2	98.46	98.46	1310.4	886.14	111.6	98.94	98.94	1339.2	890.46
2	102.3	99.61	99.61	1227.6	896.49	102.9	99.73	99.73	1234.8	897.57
3	59.2	48.46	96.92	710.4	436.14	61.6	48.94	97.88	739.2	440.46
4	52.3	49.61	99.22	627.6	446.49	52.9	49.73	99.46	634.8	447.57
5	119.1	99.71	99.71	357.3	897.39	120.8	99.79	99.79	362.4	898.11
6	104.8	99.92	99.92	314.4	899.28	105.2	99.94	99.94	315.6	899.46
7	69.1	49.71	99.42	207.3	447.39	70.8	49.79	99.58	212.4	448.11
8	54.8	49.92	99.84	164.4	449.28	55.2	49.94	97.88	165.6	449.46
9	53.6	53.55	53.55	643.2	107.10	91.9	90.11	90.11	1102.8	180.22
10	88.4	88.38	88.38	1060.8	176.76	97.9	97.52	97.52	1174.8	195.04
11	3.6	3.55	7.10	43.2	7.10	41.9	40.11	80.22	502.8	80.22
12	38.4	38.38	76.76	460.8	76.76	47.9	47.52	95.04	574.8	95.04
13	108.1	98.21	98.21	324.3	196.42	110.6	98.76	98.76	331.8	197.52
14	102.0	99.55	99.55	306.0	199.10	102.7	99.69	99.69	308.1	199.38
15	58.1	48.21	96.42	174.3	96.42	60.6	48.76	97.52	181.8	97.52
16	52.0	49.55	99.10	156.0	99.10	52.7	49.69	99.38	158.1	99.38

	$\lambda = .0833$					$\lambda = .0416$				
	$\Sigma c_i z_i = 9715$			$\Sigma M_i s(z_i) = 6555$		$\Sigma c_i z_i = 10152$			$\Sigma M_i s(z_i) = 6582$	
i	z_i	$s(z_i)$	$s(z_i)/E[X]$	$c_i z_i$	$M_i s(z_i)$	z_i	$s(z_i)$	$s(z_i)/E[X]$	$c_i z_i$	$M_i s(z_i)$
1	114.6	99.35	99.35	1375.2	894.15	119.1	99.71	99.71	1429.2	897.39
2	103.6	99.83	99.83	1243.2	898.47	104.7	99.92	99.92	1256.4	899.28
3	64.6	49.35	98.70	775.2	444.15	69.1	49.71	99.42	829.2	447.39
4	53.6	49.83	99.66	643.2	448.47	54.7	49.92	99.84	656.4	449.28
5	122.9	99.87	99.87	368.7	898.83	126.4	99.94	99.94	379.2	899.46
6	105.7	99.96	99.96	317.1	899.64	106.6	99.98	99.98	319.8	899.82
7	72.9	49.87	99.74	218.7	448.83	76.4	49.94	99.88	229.2	449.46
8	55.7	49.96	99.92	167.1	449.64	56.6	49.98	99.96	169.8	449.82
9	100.0	95.21	95.21	1200.0	190.42	108.0	98.21	98.21	1296.0	196.42
10	100.0	98.80	98.80	1200.0	197.60	102.0	99.55	99.55	1224.0	199.10
11	50.0	45.21	90.42	600.0	90.42	58.0	48.21	96.42	696.0	96.42
12	50.0	48.80	97.60	600.0	97.60	52.0	49.55	99.10	624.0	99.10
13	113.8	99.25	99.25	341.4	198.50	118.4	99.67	99.67	355.2	199.34
14	103.4	99.81	99.81	310.2	199.62	104.6	99.91	99.91	313.8	199.82
15	63.8	49.25	98.50	191.4	98.50	68.4	49.67	99.34	205.2	99.34
16	53.4	49.81	99.62	160.2	99.62	54.6	49.91	99.82	163.8	99.82

for various total cube requirements to be due to variability of demand (σ).

Figure 3 supports the argument that when a large total cube is available, items with high values of σ will be stocked in larger amounts than items with small σ. However, when the total cube available becomes more restricted, the stock of items with high variability of demand is reduced rapidly. Plots for other items which differ only in their value of σ show a similiar trend.

Reference to Figure 4 indicates that as total available cube increases, the total essentiality of the load list increases also, but at a decreasing rate. Planners who wish to increase the spare parts capacity of a tender would have to weigh the cost of increasing capacity against the value of the added cargo. Use of a computerized version of the model with a large number of items would provide a new optimal load list for the proposed increased capacity. Reference to a curve similiar to that shown in Figure 4 would be an aid to the planner who must determine whether the essentiality of the added inventory is worth the cost of increasing tender spare parts capacity.

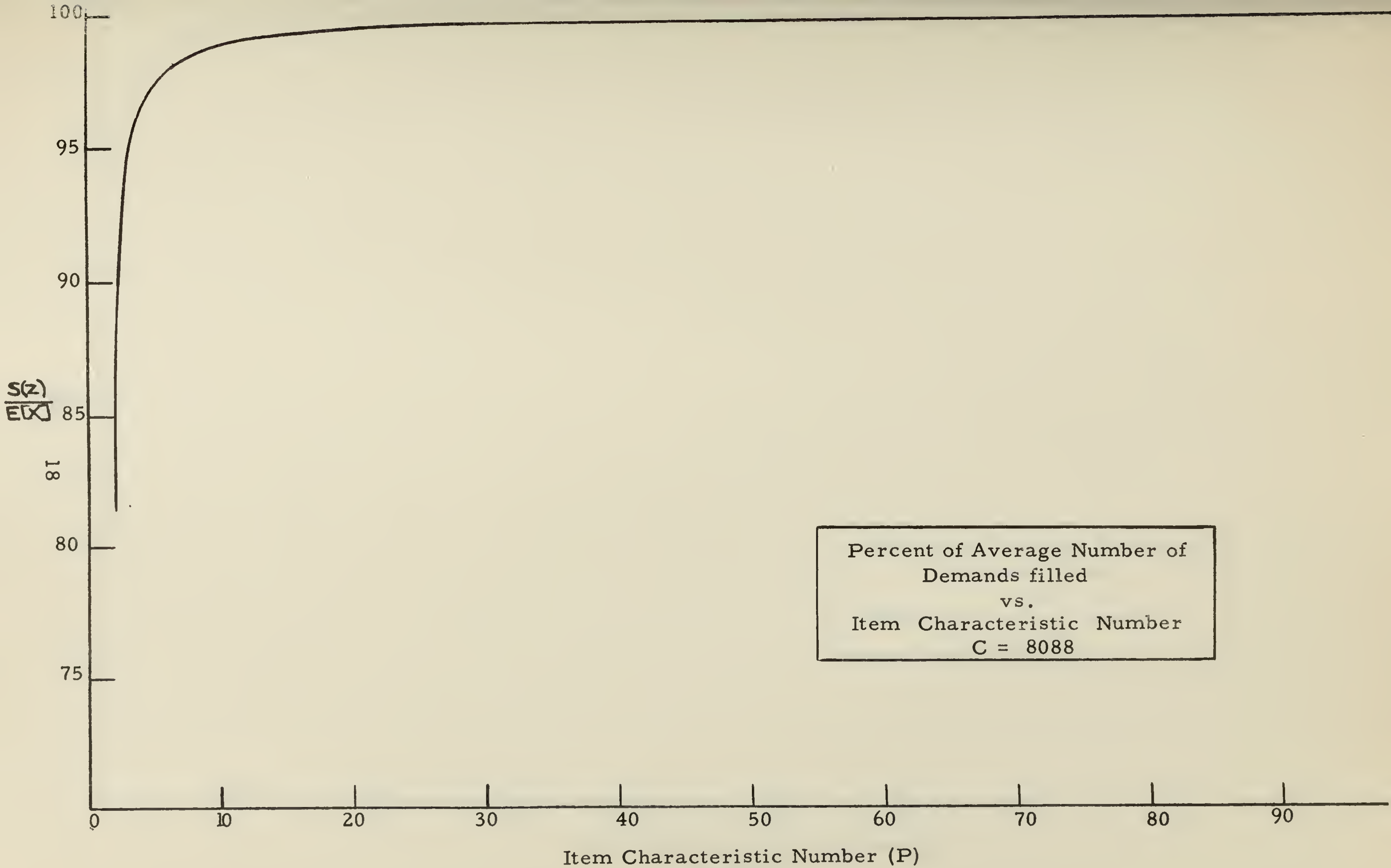

100
95
90
85
80
75
$\frac{S(z)}{E[X]}$
0
10
20
30
40
50
60
70
80
90
Percent of Average Number of
Demands filled
vs.
Item Characteristic Number
C = 8088
Item Characteristic Number (P)

Fig 2(a)

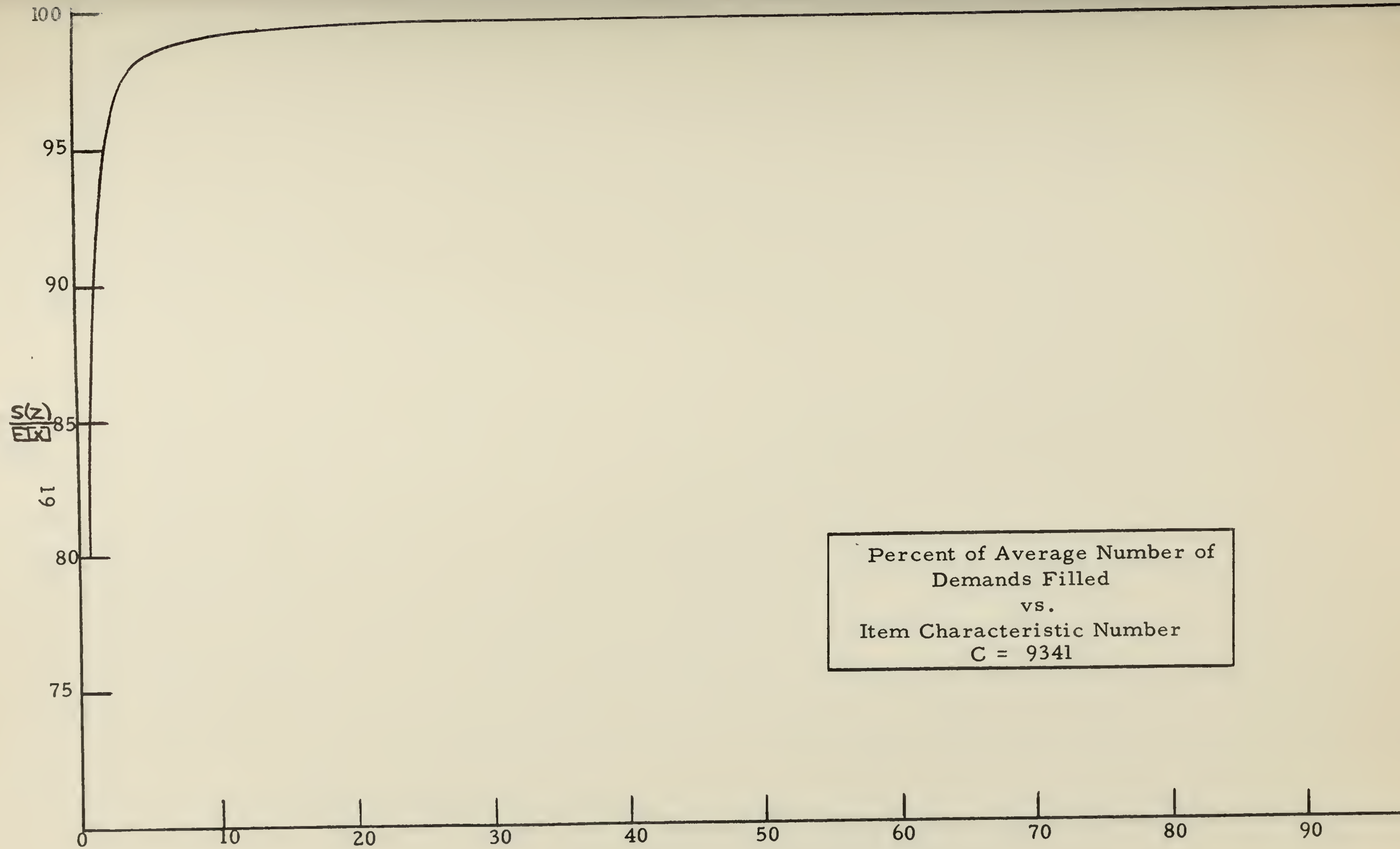
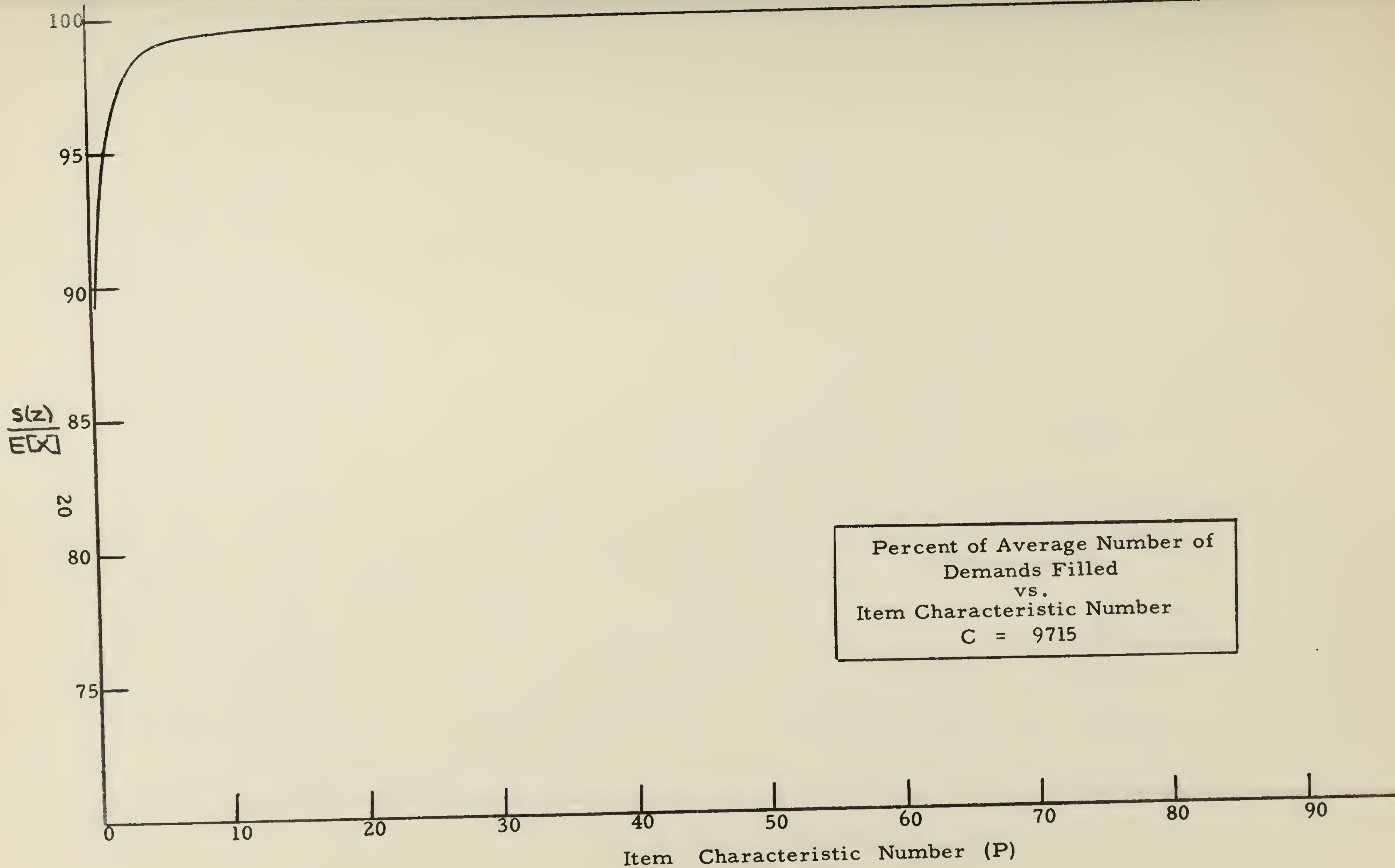

100
95
90
85
80
75
$\frac{S(z)}{E[x]}$
Percent of Average Number of
Demands Filled
vs.
Item Characteristic Number
C = 9341
0
10
20
30
40
50
60
70
80
90
Item Characteristic Number (P)

Fig 2(b)


100
95
90
S(z)/E[X] 85
80
75
0
10
20
30
40
50
60
70
80
90
Item Characteristic Number (P)
Percent of Average Number of
Demands Filled
vs.
Item Characteristic Number
C = 9715
Fig 2(c)

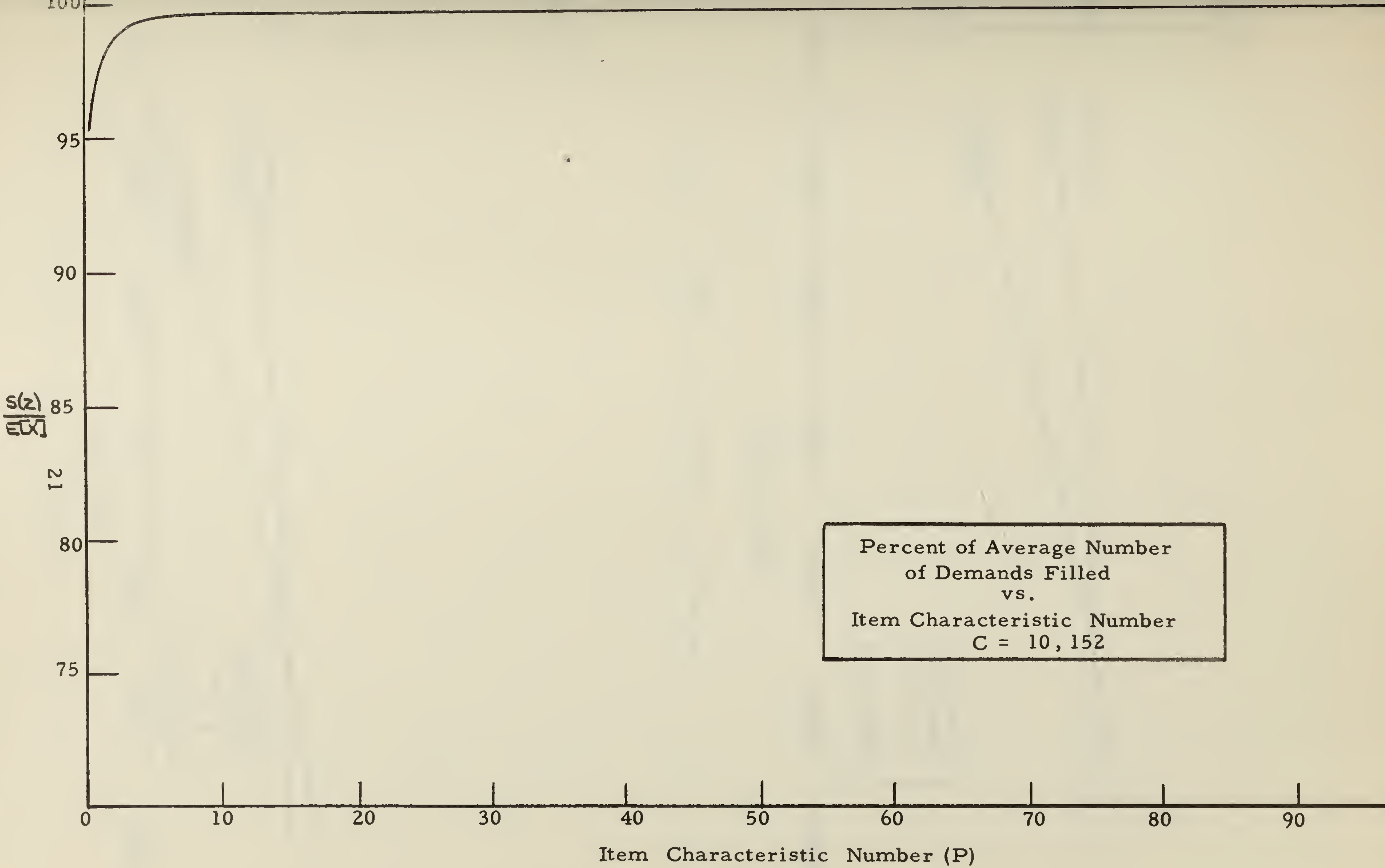

100
95
90
$\frac{S(z)}{E[X]}$ 85
80
75
0
10
20
30
40
50
60
70
80
90
Item Characteristic Number (P)
Percent of Average Number
of Demands Filled
vs.
Item Characteristic Number
C = 10, 152

Fig 2(d)

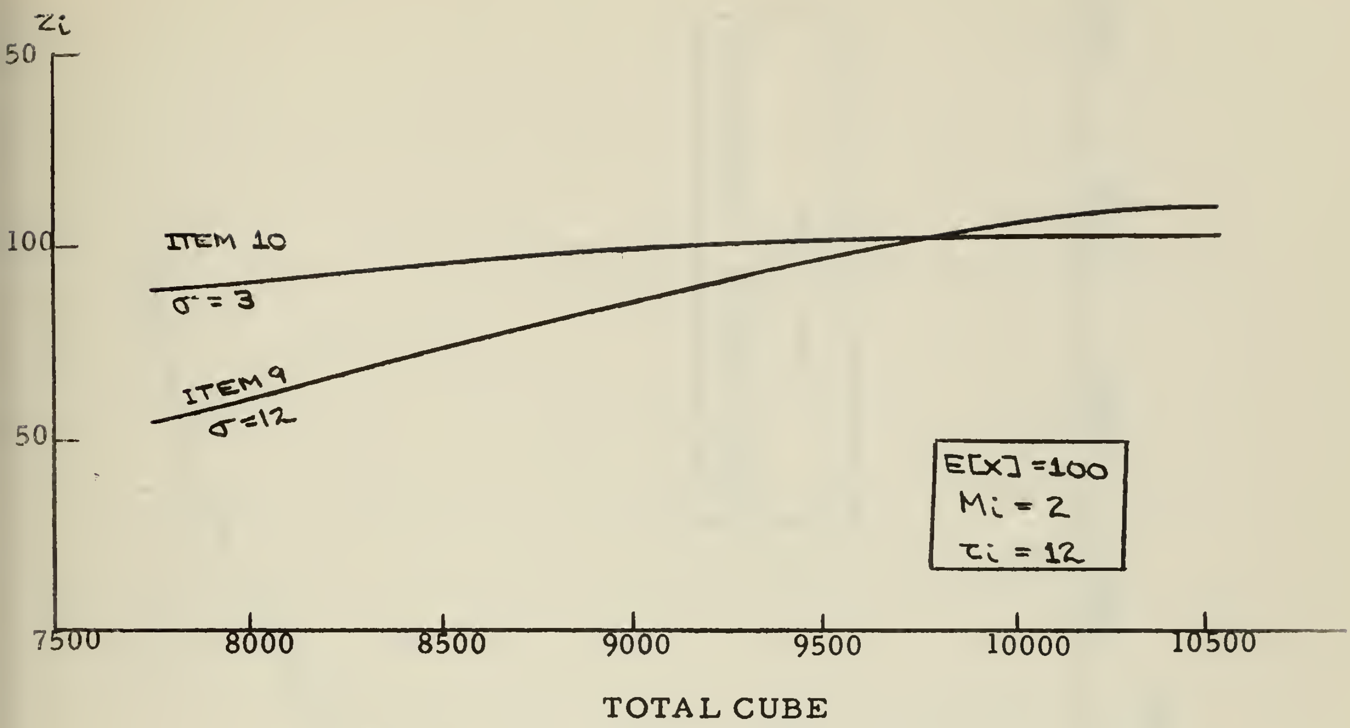
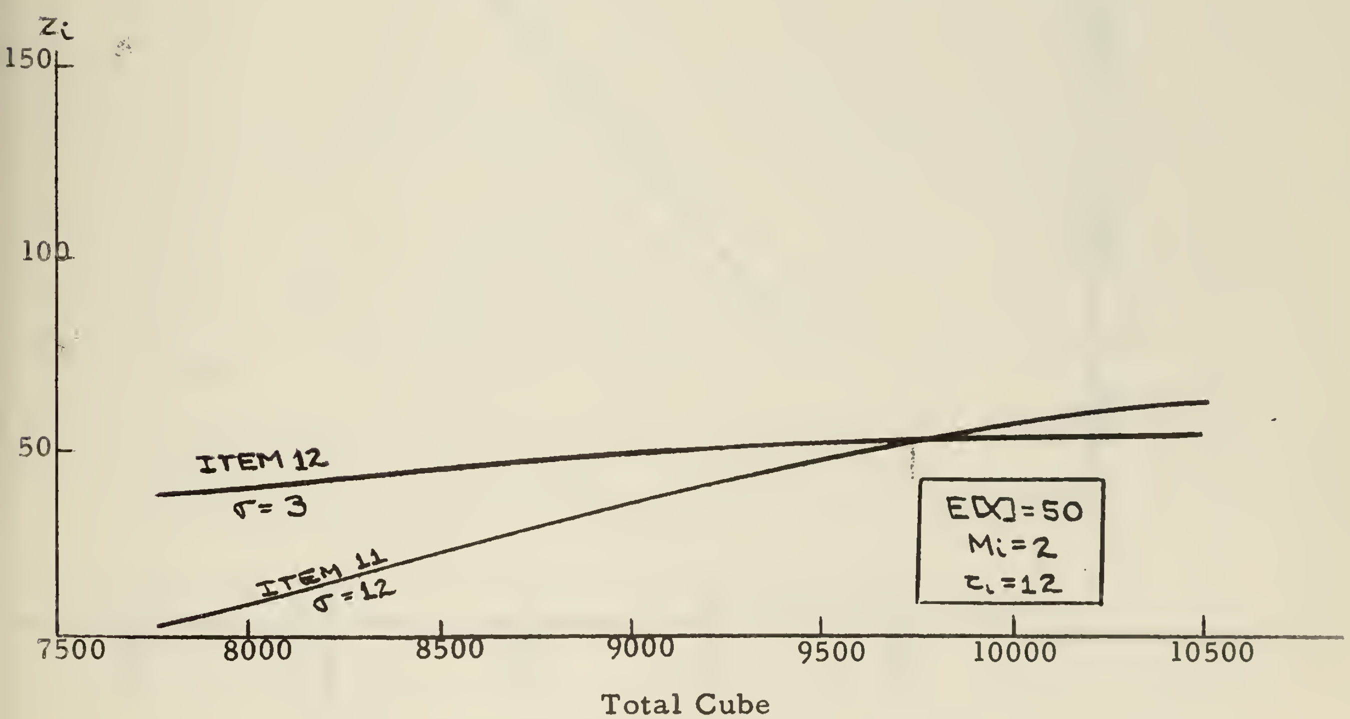


Fig 3: Effect of Variability of Demand



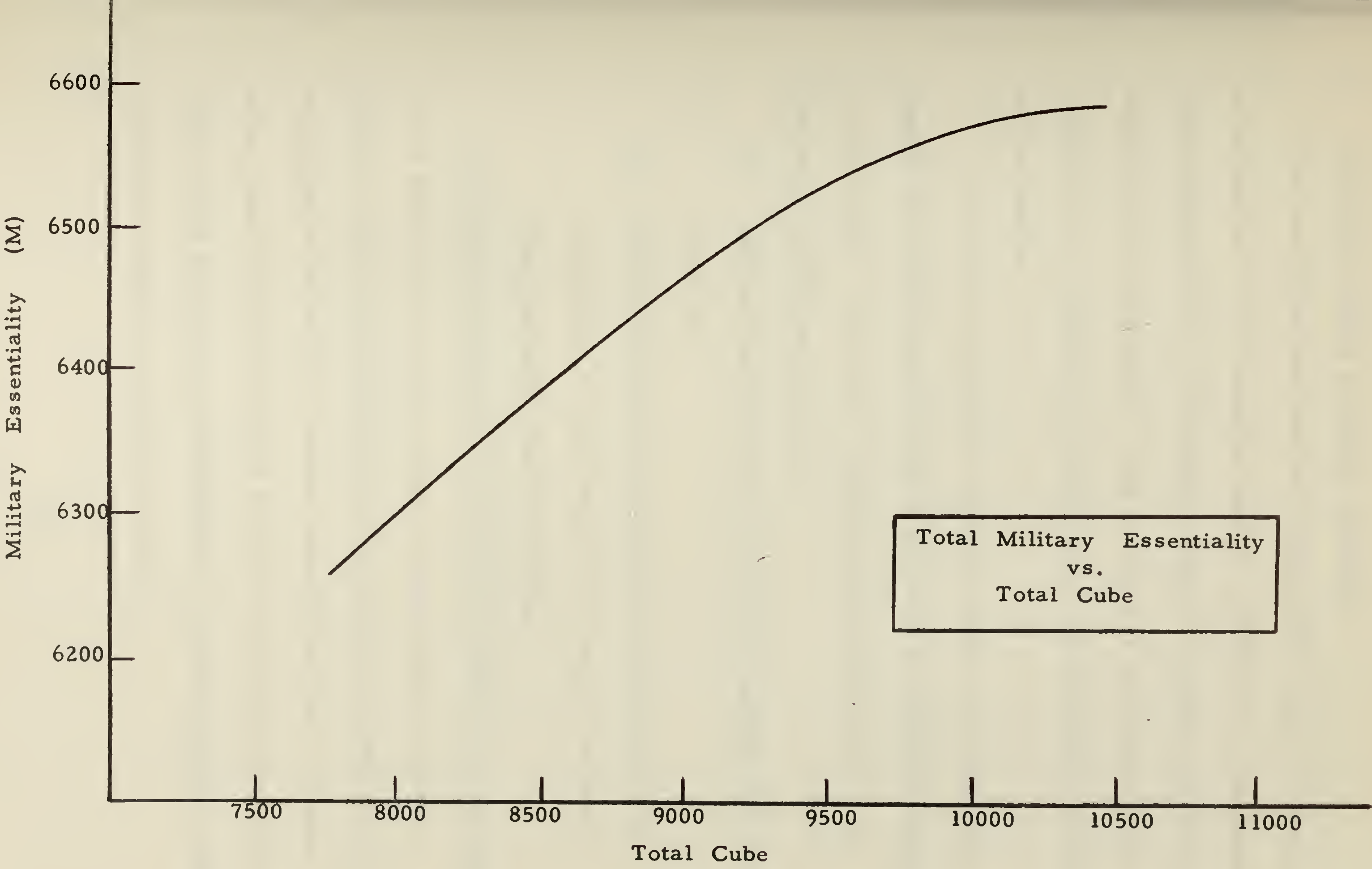

Total Military Essentiality
vs.
Total Cube
Military Essentiality (M)
6600
6500
6400
6300
6200
7500
8000
8500
9000
9500
10000
10500
11000
Total Cube

Fig 4

6. POLARIS Tender Re-supply (General)

The maximum deterrent value of the FBM weapon system can be realized only if the maximum number of SSB(N)'s are maintained in the designated areas. The logistic support concept is based on satisfying this requirement. Lack of essential spare parts can be interpreted as a deficiency in the number of ready missiles scheduled to be on station.

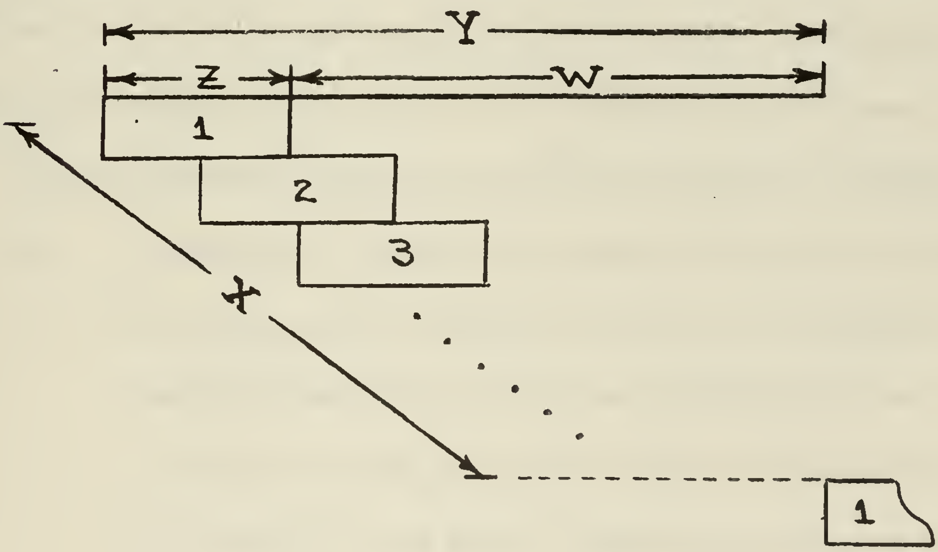
A load list is developed to allow the tender to provide effective support for a given period of time in the event re-supply is cut off for any reason. As a result, replenishment of the tender is influenced by the following factors:

(1) maintain stock levels at or near the optimum load list point, and

(2) provide delivery of nonstandard demands (items not included on the load list for one reason or another) from the stock point to the tender.

The factors considered in selecting the mode of transportation (air or surface) for an item include the transit time available, priority, physical characteristics of the material, capability and availability of the various carriers, security and cost. Of these factors those necessary to effect delivery to meet program requirements should govern and considerations of cost and equitable traffic distribution should be regarded as secondary.

In order to illustrate the considerations mentioned above consider the case where a tender is assigned to support X SSB(N)'s. The load list for the tender is designed to provide Y days support during which time each SSB(N) is alongside for Z days. (Fig 5). Y can be interpreted as one cycle during which an SSB(N) has an upkeep period of Z days and a patrol of W days (Y = Z+W). If the assumption is made that the alongside availability period for each SSB(N) is the same, then $\frac{XZ}{Y} = K$ is the number of SSB(N)'s at the tender at any time. Under this assumption an SSB(N) departs and a new one arrives every $\frac{Y}{X}$ days. When demand by each SSB(N) is assumed to be uniform, stock levels aboard the tender are depleted $\frac{1}{X}\%$ every $\frac{Y}{X}$ days. Theoretically, for the case where there is no re-supply, the tender is completely depleted of spare parts at the end of Y days.



Tender Support Cycle
Fig. 5

7. Routine Re-supply

Consider the first factor influencing re-supply mentioned previously. The average percentage of the optimum amount (z_i) which the tender maintains will be defined as the effective inventory level. This level is a function of demand, re-order policy, and the lead time associated with the transportation of the item. Consider the case where demand for all items is uniform over a ninety-day period. Fig (6) is a plot of effective inventory levels at the tender versus lead time required for delivery as a function of re-order policy.

The graph can be utilized in several different ways. If a re-order policy is chosen the graph indicates the inventory level that can be maintained for various items as a function of lead time. If lead times are fixed and a certain inventory level is desired to be maintained then the re-order policy can be obtained from the graph.

It should be noted that due to the capacity of the ship and the lead time required for delivery of an item the tender will never average an amount equal to the load list quantity for that item.

Several examples will serve to illustrate the use of Fig (6).

(a) Consider the case where an effective inventory level of 90 per cent is desired for an item. The graph indicates that this level can be maintained by re-ordering when the stock level drops to 90% provided the lead time is 10 days. Another possibility is to re-order when stock levels drop to 95% if lead time is 16 days. The re-order quantity in each case is the demand to date plus expected demand over the lead time.

(b) If the lead time for delivery of an item is fixed then the re-order policy determines the effectiveness levels maintained. With a lead time of 12 days a 60% policy results in an effective level of 76%, an 80% policy results in a level of 84.5%, a 90% policy in an 89% level and a 95% re-order policy in a level of 93%.

If the lead time is 30 days, which is analogous to that of surface re-supply by AK, then the maximum possible effectiveness level is 83% for the example shown.

It is obvious that high inventory levels can be maintained effectively only for the case when the lead times are short. This illustrates the requirement for air transportation of spare parts as opposed to surface re-supply.

In addition to the effect on effective inventory levels maintained, air re-supply has the advantage of reducing the confusion factor at the tender when supplies are received. Rather than being saturated with parts delivered by an AK all at one time the tender is able to maintain better control over spare parts storage. This in turn tends to reduce "loss" of items due to misplacement.

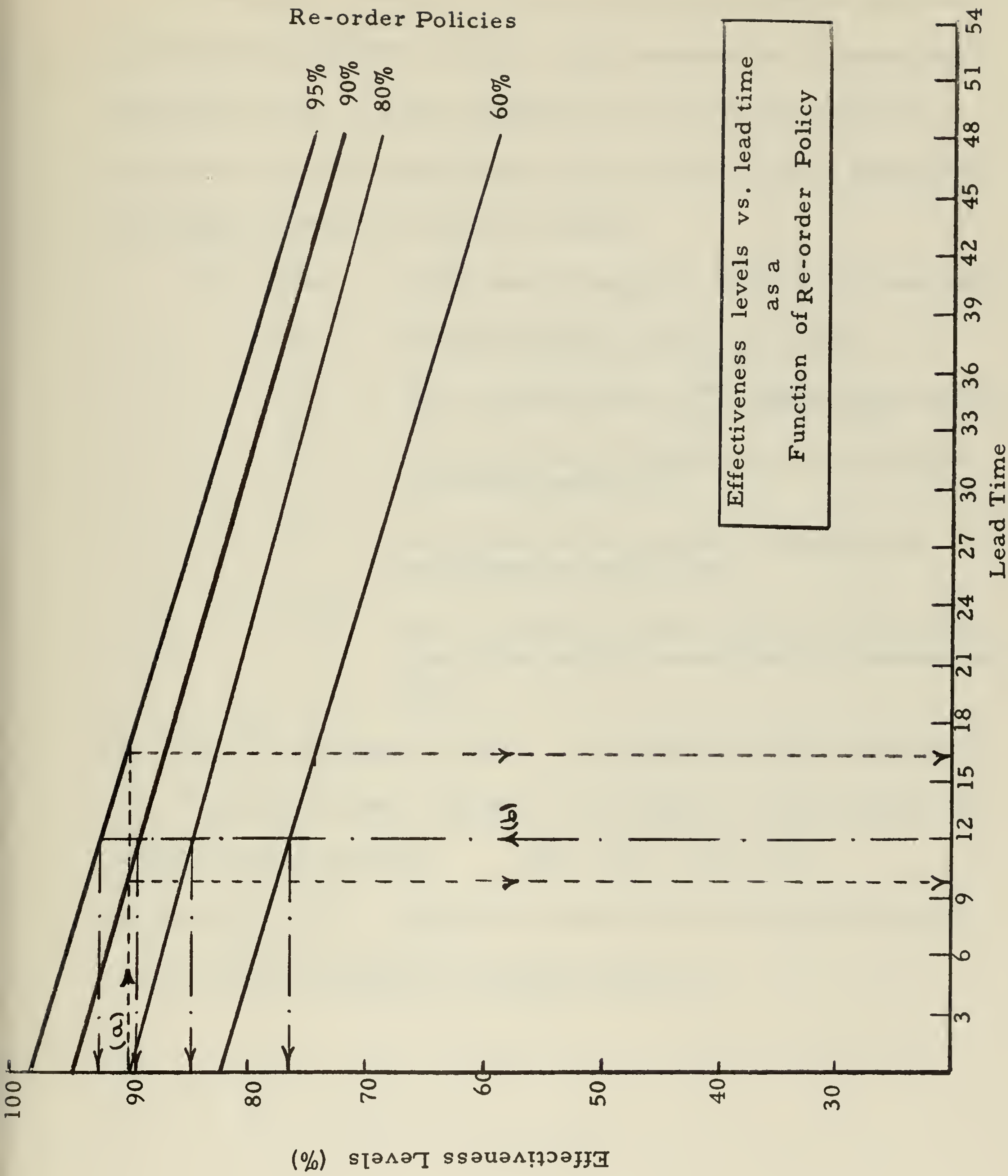

Re-order Policies
95%
90%
80%
60%
Effectiveness levels vs. lead time as a Function of Re-order Policy
(a)
(b)
Lead Time
3 6 9 12 15 18 21 24 27 30 33 36 39 42 45 48 51 54
Effectiveness Levels (%)
100 90 80 70 60 50 40 30

Fig. 6

8. Determination of resupply method for NIS/Nonstandard items.

Consider a spare part that is required by an SSB(N) and that is not in stock (NIS) on the tender. If it is assumed that a spare part NIS on the tender is always available at the stock point then the mode of re-supply transportation (air or surface) can be determined by using a criterion developed as follows:

Let, D_{AK} = Julian date of departure of AK from stock point.

T_{AK} = Days AK spends in transit to tender.

S_j = Days remaining until j^{th} SSB(N) starts patrol.

R_j = Julian day that j^{th} SSB(N) generates demand for an NIS spare part.

C = Days required for loading, unloading and installation of spare part.

LT = Days required by spare part to travel by air from stock point to tender (MATS, commercial, etc.).

If $R_j + S_j \geq D_{AK} + T_{AK} + C$, re-supply the spare part by AK;

If $R_j + S_j < D_{AK} + T_{AK} + C$, re-supply the spare part by air, subject to the constraint $R_j + S_j \geq LT$.

If $R_j + S_j < LT$, then it is impossible to re-supply the spare part prior to the SSB(N)'s scheduled departure.

9. Conclusions

1. The number of stockouts will be reduced if stocks are maintained in accordance with the model. The model maximizes the average number of demands filled over some length of time. This is equivalent to minimizing the number of stockouts over the same length of time.

2. As the spare parts stowage capacity of the tender is increased, the total worth (military essentiality) of the inventory is increased, but at a decreasing rate. For example, if a small tender and a large tender each have their cubic capacity increased by the same constant amount, the small tender has a larger percentage gain in the military essentiality of cargo than the large tender.

3. Items with high essentiality to cube ratios are generally carried at high stock levels. For those items with nearly the same pattern of demand ($\frac{m}{\sigma}$), the model maintains high inventories of high essentiality, low cube items, and low inventories at low essentiality, high cube items.

4. When items must be deleted from inventory because of space limitations, those with the lowest essentiality to cube ratio are the first to be eliminated.

5. The spare parts stowage capacity of the tender is the limiting factor in determining the length of the period over which

demands filled should be maximized. The load list should be designed to maximize demands filled over a given length of time.

6. Re-supply by air is required to maintain high inventory levels of essential spare parts aboard the tender. In order to maintain essential spare parts inventories close to the optimum load list level at all times, it is necessary to have short re-supply lead times, or a large tender capacity. Since tender capacity is limited, high inventories can only be achieved by utilizing the short lead times associated with air re-supply.

7. Re-supply ships cannot meet demands for non-standard items. The long lead time associated with ship re-supply precludes the possibility of meeting demands for non-standard items as they occur. For example, it is a rare occurrence when a submarine can execute an order for a non-standard item and have it delivered by ship re-supply <u>before</u> going on patrol again.

8. The iterative nature of the model and the large number of items in a real life inventory make the use of a computer mandatory.

APPENDIX I

Derivation of Optimal Tender Load List Model

(1) $$E[x_i] = \int_0^\infty x_i f_i(x_i)\,dx_i .$$

(2) $$b_i(z_i) = \int_{z_i}^\infty (x_i - z_i) f_i(x_i)\,dx_i .$$

(3) $$s_i(z_i) = \int_0^{z_i} x_i f_i(x_i)\,dx_i + z_i \int_{z_i}^\infty f_i(x_i)\,dx_i .$$

(4) $$E[x_i] = s_i(z_i) + b_i(z_i) .$$

Essentiality of a spare part is an important factor in deciding on a load list which will maximize the average number of demands filled by the tender. On the basis of past usage data the model should indicate the quantity of each item to be stocked such that the total essentiality function M is a maximum subject to the tender stowage limitations.

The function $g_i(x_i, z_i)$ will be introduced to represent a utility valve associated with the ability of the tender to meet demand for an item.

$$g_i(x_i, z_i) = \begin{cases} M_i x_i & \text{when } x_i \le z_i \\ M_i z_i & \text{when } x_i \ge z_i \end{cases} \qquad i = 1, 2 \ldots\ldots N$$

It should be apparent that if demand is less than what is stocked the tender is able to provide 100% service. On the other hand if demand exceeds the stock level then the tender does as best as it can by providing an amount z_i .

In accordance with the measure of effectiveness mentioned previously the objective is to maximize the total utility value as defined above

$$\text{Max.} \quad M(z_1, z_2 \ldots z_N) = E\left[\sum_{i=1}^{N} g_i(x_i, z_i)\right]$$

subject to the constraint

$$\sum_i c_i z_i = C \quad , \quad z_i > 0 .$$

$$E\left[\sum_i g_i(x_i, z_i)\right] = \int \cdots \int \left[\sum_i g_i(x_i, z_i)\right] f(x_1, x_2 \ldots x_N)\, dx_1 \ldots dx_N$$

where $f(x_1, x_2 \ldots\ldots x_N)$ is the joint probability function of $x_1, x_2, \ldots\ldots x_N$.

By assumption (7) this joint probablity density function can be written as:

$$f(x_1, x_2 \ldots\ldots x_N) = f_1(x_1) f_2(x_2) \ldots\ldots f_N(x_N)$$

[By definition $\int_0^\infty f_i(x_i)\, dx_i = 1$]

As a result $E\left[\sum_i g_i(x_i, z_i)\right]$ can be expressed as follows:

$$E\left[\sum_i g_i(x_i, z_i)\right] = \int_0^\infty g_1(x_1, z_1) f_1(x_1)\, dx_1 + \ldots \int_0^\infty g_N(x_N, z_N) f_N(x_N)\, dx_N .$$

Let

$$M_i s_i(z_i) = E\left[g_i(x_i, z_i)\right]$$

$$\therefore \; M_i s_i(z_i) = \int_0^\infty g_i(x_i, z_i) f_i(x_i)\, dx_i$$

Maximize $\quad M(z_1, z_2 \ldots z_N) = \sum_i M_i s_i(z_i)$

Subject to: $\quad \sum_i c_i z_i = C$

The problem presented is one of maximizing a function of n variables where the variables are related by a single constraint equation. The problem could possibly be solved by eliminating some of the variables using the constraint equation. Eventually the problem could probably be reduced to an ordinary maximization problem. This procedure is not always feasible and therefore the technique of Lagrangian multipliers will be utilized.

To find the points of $M = M(z_1, z_2 \ldots\ldots z_N)$ where the variables $z_1, z_2 \ldots z_N$ are related by the equation $\sum_i^N c_i z_i - C = 0$ the function L is defined as follows:

$$L = \sum_{i=1}^{N} M_i s_i(z_i) - \lambda\left[\sum_{i=1}^{N} c_i z_i - C\right]$$

where λ is a Lagrangian multiplier

$$0 = \frac{\partial L}{\partial z_i} = \frac{d}{dz_i}\left(M_i s_i(z_i) - \lambda[c_i z_i - C]\right)$$

$$= \frac{d}{dz_i}[M_i s_i(z_i)] - \lambda c_i, \quad i = 1, 2, \ldots N$$

As a result, the following set of $(n + 1)$ equations in $(n + 1)$ variables $z_1, z_2 \ldots z_N$ and λ is obtained.

$$\frac{\partial M}{\partial z_1} - \lambda \frac{\partial}{\partial z_1}(c_1 z_1 - C) = 0$$

$$\frac{\partial M}{\partial z_2} - \lambda \frac{\partial}{\partial z_2}(c_2 z_2 - C) = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\frac{\partial M}{\partial z_N} - \lambda \frac{\partial}{\partial z_N}(c_N z_N - C) = 0$$

$$\sum_{i=1}^{N} c_i z_i - C = 0$$

By rearranging the terms the above equations can be put in the form

$$\frac{d}{dz_i}[M_i s_i(z_i)] = \lambda c_i$$

(6) $$\frac{d}{dz_i} s_i(z_i) = \lambda\left(\frac{c_i}{M_i}\right)$$

Recalling equation (3)

$$(6') \quad \frac{d}{dz_i}\left[\int_0^{z_i} x_i f_i(x_i)dx_i + z_i\int_{z_i}^{\infty} f_i(x_i)dx_i\right] = \lambda\left(\frac{c_i}{M_i}\right)$$

Lebnitz's rule is utilized to evaluate the left hand side of this equation since the units of integration depend on the parameter z_i.

Leibnitz's Rule: $$\frac{d}{dt}\int_{a(t)}^{b(t)} h(x,t)dt = h[b(t),t]\,b'(t) - h[a(t),t]\,a'(t) + \int_{a(t)}^{b(t)} \frac{\partial h}{\partial t}(x,t)\,dx$$

$$\therefore \frac{d}{dz_i}\left[\int_0^{z_i} x_i f_i(x_i)dx_i + z_i\int_{z_i}^{\infty} f_i(x_i)dx_i\right]$$
$$= [z_i f_i(z_i) - 0 + 0] + [0 - z_i f_i(z_i)] + \int_{z_i}^{\infty} f_i(x_i)dx_i$$
$$= \int_{z_i}^{\infty} f_i(x_i)dx_i$$

Since $\int_0^{\infty} f_i(x_i)dx_i = 1 = \int_0^{z_i} f_i(x_i)dx_i + \int_{z_i}^{\infty} f_i(x_i)dx_i$

Then $\int_{z_i}^{\infty} f_i(x_i)dx_i = 1 - \int_0^{z_i} f_i(x_i)dx_i$

By substituting and rearranging terms equation (6') reduces to the following:

$$1 - \int_0^{z_i} f_i(x_i)dx_i = \lambda\left(\frac{c_i}{M_i}\right)$$

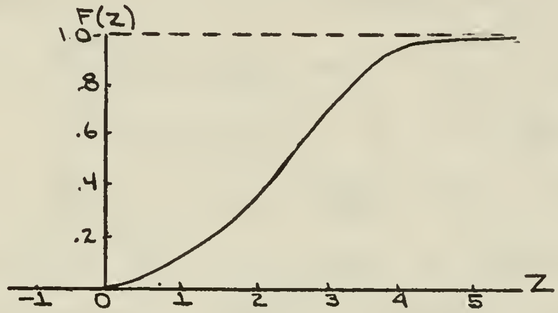
or

$$\int_0^{z_i} f_i(x_i)dx_i = 1 - \lambda\left(\frac{c_i}{M_i}\right) \qquad (5)$$

Equation (5), subject to the constraint equation, provides the method where-by the set Z can be determined. The two equations determine a unique set of z_i's provided that for all i, $f_i(x_i) > 0$ for all values of x_i.

The distribution function, $F(\cdot)$, of a numerical valued random phenomenon is defined as having as its value, at any real number z , the probability that an observed value of the random phenomenon will be less than or equal to the number z . ($F(z) = P[z' \leq z]$). Since the left side of (5) is a distribution function the following condition applies:

$$0 \leq \int_0^{z_i} f_i(x_i)\,dx_i \leq 1 .$$



In order to solve for z_i in equation (5) we must assume a value for λ . It would materially decrease the iterative requirements if the possible values of λ could be confined to a portion of the real line.

Since $\quad 0 \leq \int_0^{z_i} f_i(x_i)\,dx_i \leq 1$

Then $\quad 0 \leq 1 - \lambda\left(\frac{c_i}{M_i}\right)$

$$-1 \leq \lambda\left(\frac{c_i}{M_i}\right) \Rightarrow \frac{M_i}{c_i} \geq \lambda$$

$$1 - \lambda\left(\frac{c_i}{M_i}\right) \leq 1$$

$$-\lambda\left(\frac{c_i}{M_i}\right) \leq 0 \Rightarrow \lambda \geq 0$$

$$\therefore \quad \left(\frac{M_i}{c_i}\right) \geq \lambda \geq 0$$

These limits apply for all i and therefore λ must fall in the range . $\min\left(\frac{M_i}{c_i}\right) \geq \lambda \geq 0$ in order for all items to be included in the load list.

Derivation of $s_i(z_i)$ for the case where demand follows the normal distribution:

$$s_i(z_i) = \int_0^{z_i} x_i f_i(x_i)\,dx_i + z_i \int_{z_i}^{\infty} f_i(x_i)\,dx_i$$

where $$f_i(x_i) = \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2}$$

$$\int_0^{\infty} f_i(x_i)\,dx_i = 1 = \int_0^{z_i} f_i(x_i)\,dx_i + \int_{z_i}^{\infty} f_i(x_i)\,dx_i$$

$$\therefore \int_{z_i}^{\infty} f_i(x_i)\,dx_i = 1 - \int_0^{z_i} f_i(x_i)\,dx_i$$

but $$\int_c^{z_i} f_i(x_i)\,dx_i = 1 - \lambda\left(\frac{c_i}{M_i}\right)$$

$$\therefore \int_{z_i}^{\infty} f_i(x_i)\,dx_i = \left\{1 - \left[1 - \lambda\left(\frac{c_i}{M_i}\right)\right]\right\}$$

$$\therefore s_i(z_i) = \int_0^{z_i} x_i f_i(x_i)\,dx_i + z_i\left[1 - \left(1 - \lambda \frac{c_i}{M_i}\right)\right]$$

In order to solve for $s_i(z_i)$ an explicit expression must be found for $\int_0^{z_i} x_i f_i(x_i)\,dx_i$. The procedure is as follows:

$$\text{(II-1)} \quad \int_0^{z_i} x_i f_i(x_i)\,dx_i = \int_0^{z} (x_i) \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2} dx_i$$

By taking the derivative of $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2}$ the following expression is obtained:

$$\frac{d}{dx_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2} \left(\frac{x_i - m_i}{\sigma_i}\right) \frac{1}{\sigma_i}$$

$$= -\frac{(x_i - m_i)}{\sqrt{2\pi}\,\sigma_i^2} e^{-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2}$$

$$= -\frac{x_i}{\sqrt{2\pi}\,\sigma_i^2} e^{-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2} + \frac{m_i}{\sqrt{2\pi}\,\sigma_i^2} e^{-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i^2}\right)}$$

Integrating both sides of the equation between the limits of 0 and z_i :

$$\int_0^{z_i} \frac{d}{dx_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2} = -\int_0^{z_i} \frac{x_i}{\sqrt{2\pi}\,\sigma_i^2} e^{-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2} dx_i + \int_0^{z_i} \frac{m_i}{\sqrt{2\pi}\,\sigma_i^2} e^{-\frac{1}{2}\left(\frac{x_i - m_i}{\sigma_i}\right)^2} dx_i$$

Rearranging terms:

$$\int_0^{z_i} \frac{x_i}{\sqrt{2\pi}\sigma_i^2} e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2} dx = \int_0^{z_i} \frac{m_i}{\sqrt{2\pi}\sigma_i^2} e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2} dx_i - \int_0^{z_i} \frac{d}{dx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2}$$

Note that the expression on the left is the same as equation II-1 except for a factor σ_i. In order to obtain the same expression multiply both sides by σ_i which results in

$$\int_0^{z_i} \frac{x_i}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2} dx_i = \int_0^{z_i} \frac{m_i}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2} dx - \int_0^{z_i} \frac{d}{dx_i} \frac{\sigma_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2}$$

but $$-\int_0^{z_i} \frac{d}{dx_i} \frac{\sigma_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2} = -\frac{\sigma_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2} \Bigg|_0^{z_i}$$

$$= -\frac{\sigma_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z_i-m_i}{\sigma_i}\right)^2} + \frac{\sigma_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{m_i}{\sigma_i}\right)^2}$$

Observe that

$$\int_0^{z_i} \frac{m_i}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2} dx = m_i \int_0^{z_i} \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2}$$

but $$\int_0^{z_i} f_i(x_i)\, dx_i = \left[1 - \lambda\left(\frac{c_i}{M_i}\right)\right]$$

$$\therefore\ m_i \int_0^{z_i} \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}\left(\frac{x_i-m_i}{\sigma_i}\right)^2} dx_i = m_i\left[1-\lambda\left(\frac{c_i}{M_i}\right)\right]$$

Combining the above derived relationships an explicit equation for $S_i(z_i)$ results:

$$(3') \quad S_i(z_i) = m_i\left[1-\lambda\left(\frac{c_i}{M_i}\right)\right] + z_i\left[1-\left(1-\lambda\frac{c_i}{M_i}\right)\right] + \frac{\sigma_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{m_i}{\sigma_i}\right)^2} - \frac{\sigma_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z_i-m_i}{\sigma_i}\right)^2}$$

for $z_i > 0$

APPENDIX III

A computer program based on the flow diagram illustrated in Figure 7 was written in the Neliac language and compiled on the United States Naval Postgraduate School CDC 1604 digital computer. The Neliac statements are found in this appendix following Figure 7.[1] Outputs for several choices of parameter values confirmed the results shown inS ection 5.

As indicated by the computer flow diagram in Figure 7 the first step in the operation of the program is to carry out the routine input procedures.

The next step is the computation of the value of lambda equal to the smallest fraction, $\left(\frac{M_i}{C_i}\right)$. Denote this minimum ratio as min $\left(\frac{M_i}{C_i}\right)$. This value of lambda is used to compute all values of AREA = $\left(1 - \lambda \frac{C_i}{M_i}\right)$ for i = 1,...N. Each z_i is computed from:

$$T_{u+1} = \frac{T_u - \frac{1}{2}\,\mathrm{erf}\left(\frac{T_u}{\sqrt{2}}\right) - \mathrm{AREA}}{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{T_u}{\sqrt{2}}\right)^2}$$

which provides an approximation of the normal distribution function. In the expression above:

$$\mathrm{erf}\left(\frac{T_u}{\sqrt{2}}\right) = 1 - \frac{1}{P^8}$$

where

$$P = 1 + 0.14112821\left(\frac{T_u}{\sqrt{2}}\right) + 0.08864027\left(\frac{T_u}{\sqrt{2}}\right)^2 + 0.02743349\left(\frac{T_u}{\sqrt{2}}\right)^3 - 0.00039446\left(\frac{T_u}{\sqrt{2}}\right)^4 + 0.00328975\left(\frac{T_u}{\sqrt{2}}\right)^5$$

---

[1]Details of the Neliac language may be obtained from Prof. R. M. Thatcher, Operations Research Department, U.S. Naval Postgraduate School, Monterey, California.

The product $c_i z_i$ is computed, stored and summed. The result, $\sum_i c_i z_i$ is stored in the cell named SUMCZ.

This sum, $\sum_i c_i z_i$, is then compared with the contents of Big C, the specified total cube available for the storage of spare parts. If SUMCZ $>$ Big C, then the cube of the load list, for the given lambda, is greater than the capacity allowed for storage. When this occurs the item with the lowest essentiality to cube ratio is discarded, a new $\lambda = \min\left(\frac{M_i}{c_i}\right)$ is computed from the remaining items and is used to determine a new SUMCZ. This new value of SUMCZ is compared with Big C. If SUMCZ remains larger than Big C the item with the lowest essentiality to cube ratio is discarded. This procedure is repeated until SUMCZ becomes equal to or less than the cube available, Big C. When a new load list is found that meets the cube constraint then the essentiality of the load list is computed and a print-out is obtained for z_i, $s(z_i)$, λ, $\sum c_i z_i$, and $\sum M_i s(z_i)$. A table of the items discarded is also provided.

If the cube of the load list computed for the initial value of lambda ($\lambda = \min \frac{M_i}{c_i}$) is less than the capacity allowed for storage, then a print-out is obtained for z_i, $s(z_i)$, λ, $\sum_i c_i z_i$ and $\sum M_i s(z_i)$. In order to find the load list that most nearly meets the cube constraint it is necessary to evaluate $\sum c_i z_i$ for several assumed values of λ.

The number of values of λ to be assumed is specified in the program by the parameter K. If, for example, K=20 then the variables z_i, $s(z_i)$, $\sum_i c_i z_i$ and $\sum_i M_i s(z_i)$ will be evaluated for twenty equally spaced values of λ between min $\frac{M_i}{c_i}$ and zero. In this case twenty print-outs will be provided. The optimum load list is taken as the load list that most nearly satisfies the cube constraint.

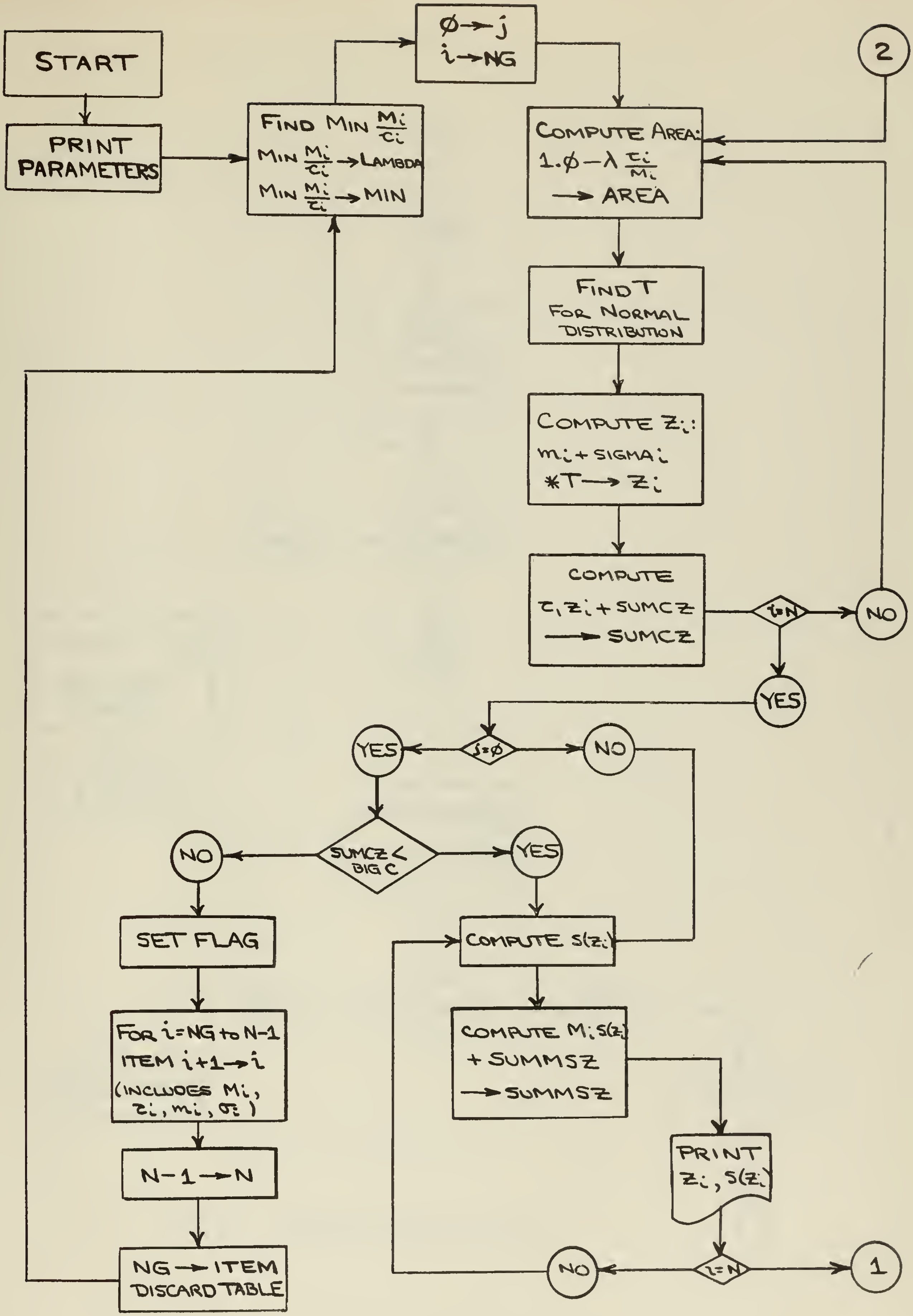


Fig 7. Computer Flow Diagram

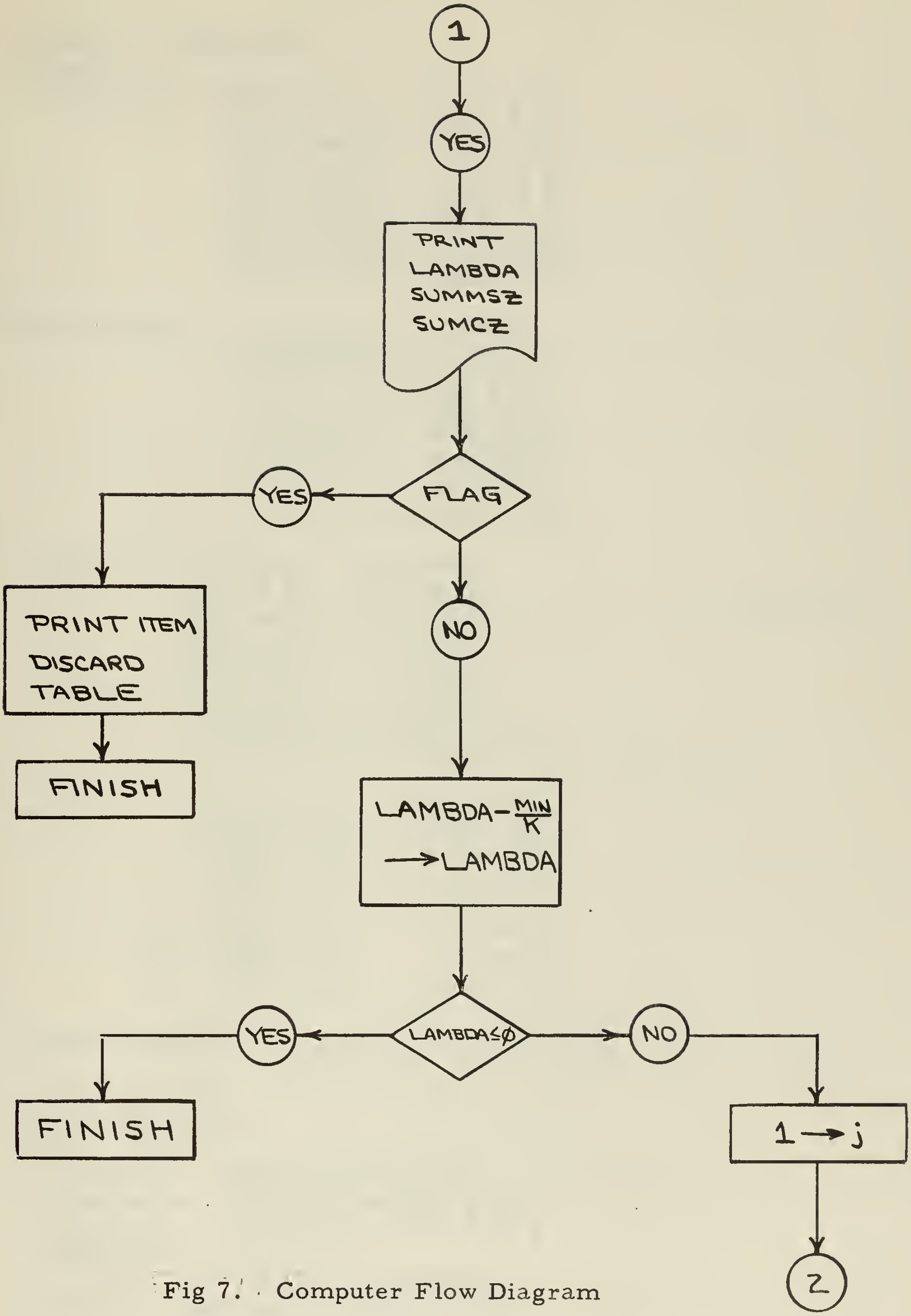


Fig 7. Computer Flow Diagram

```
BIG C = 10847*0
NN = 16,
MX (16) =  100.0 *0,  100.0 *0,
            50.0 *0,   50.0 *0,
           100.0 *0,  100.0 *0,
            50.0 *0,   50.0 *0,
           100.0 *0,  100.0 *0,
            50.0 *0,   50.0 *0,
           100.0 *0,  100.0 *0,
            50.0 *0, 0050.0 *0,
^
SIGMA(16) =  12.0 *0,   3.0 *0,
             12.0 *0,   3.0 *0,
             12.0 *0,   3.0 *0,
             12.0 *0,   3.0 *0,
             12.0 *0,   3.0 *0,
             12.0 *0,   3.0 *0,
             12.0 *0,   3.0 *0,
             12.0 *0, 0003.0 *0,
^
W(16) =  9.0 *0,   9.0 *0,
         9.0 *0,   9.0 *0,
         9.0 *0,   9.0 *0,
         9.0 *0,   9.0 *0,
         2.0 *0,   2.0 *0,
         2.0 *0,   2.0 *0,
         2.0 *0,   2.0 *0,
         2.0 *0, 0002.0 *0,
^
C(16) =  12.0 *0,   12.0 *0,
         12.0 *0,   12.0 *0,
          3.0 *0,    3.0 *0,
          3.0 *0,    3.0 *0,
         12.0 *0,   12.0 *0,
         12.0 *0,   12.0 *0,
          3.0 *0,    3.0 *0,
          3.0 *0,  0003.0 *0,
^
ITEM NR(16) = 1, 2, 3, 4, 5, 6, 7, 8, 9,
              10, 11, 12, 13, 14, 15, 16,
MIN = 0*0
KK = 20.0*0
II = 000,
LAMBDA = 0000.0000 *0,
SUMCZ = 000000.0000 *0,
SUMWSZ = 000000.0000 *0,
AREA (16) = 0*0,
Z (16) = 000000.0000 *0,
SZ(16) = 000000.0000 *0,
```

^ Indicates a blank card

9 Indicates a card of all nines

```
   WOC = 0*0,   AREAM5 = 000.0000*0,
   NEGPT5 = -.5*0
   X = 0*0, Y = 0*0, T = 000.0000*0,
   SIGØ2PI = 0*0
   EXP1 = 0*0,
   EXP2 = 0*0,
   SQRT2PI = 0*0,
   TWØPI = 6.2831853072*0,   IX = 00,
^
   MFLAG, NG, MIN FLAG,
   FLAG, JFLAG, ITEM DISCARD (16) = 000,
   T2 = 0*0, T1 = 0*0,
   NULL = 777770007777700008,
^
   ; START :, {<>;},
         ; {<< PRØGRAM | DAMICØ | HARVEY >>,,,},
           PRINT PARAM,
^
     0 → NG → FLAG → JFLAG → MFLAG → N,
     SQRTF (TWØPI ; SQRT2PI),
^
   FIND MIN WØC :
        0 → M,
   REPEAT :
        IF ITEM NR [M] = NULL : M+1 → M, M = NN :
                                   FINISH . REPEAT. ;;
        W[M] / C[M] → LAMBDA,
^
   0 → MIN FLAG,
   FØR K = 0(1) NN-1
        { IF ITEM NR [K] = NULL : DISCARD WØC. ;
          W[K] / C[K] → WØC < LAMBDA :
          WØC → LAMBDA,   K → MIN FLAG ;;
          DISCARD WØC : }
^
   IF MFLAG = 0 : LAMBDA → MIN, 1 → MFLAG ;;
^
   COMPUTE AREA :
        0 → SUMCZ,
        FØR I = 0(1) NN-1
          { IF ITEM NR [I] = NULL : ØMIT. ;
            1.0 - LAMBDA * (C[I] / W[I]) → AREA [I],
            IF AREA [I] = 0 : 0 - 3.87 → T, CØNT. ;
            IF AREA [I] < 0.5 : 0.5 - AREA [I] → AREAM5,
            AREAM5 < 0.00001 : 0 → T, CØNT. ;
                NORMEX (AREAM5 ; T),
                  0 - T → T, CØNT.
                AREA[I] - 0.5 → AREAM5,
            AREAM5 < 0.00001 : 0 → T, CONT. ;
                NORMEX (AREAM5 ; T) ;
^
   CØNT :
      I + 1 → IX,
      , { << IX | = | > IX < U LAMBDA | = |
        > LAMBDA < U AREA | = | > AREA [I] <
```

```
     U AREAM5 |
     = |> AREAM5 U <T |= |>T >},
^
  MX[I] + SIGMA[I] * T ——> Z[I],
  Z[I] * C[I] + SUMCZ ——> SUMCZ,
  ØMIT : },
^
  , {PRINT << SUMCZ |=|> SUMCZ >},
^
  IF JFLAG = 0 : TEST. CØMPUTE SZ.
^
  TEST: SUMCZ < BIGC : CØMPUTE SZ. ;
^
  1 ——> FLAG,
  NG + 1 ——> NG,
  MIN FLAG ——> L ——> ITEM DISCARD[N],
             N + 1 ——> N,
  NULL ——> ITEM NR[L],
  FIND MIN WØC.
^
  0 ——> II,
  CØMPUTE SZ : , {<>,,,},
             {<< | I UU Z UU 's(z) >>,},
        0 ——> SUMWSZ,
^
  ØN: FØR I = 0(1)NN-1 {IF ITEM NR[I] = NULL: SKIP;
         SIGMA[I] / SQRT2PI ——> SIGØ2PI,
^
        MX[I] / SIGMA[I] ——> X * X ——> X,
        NEGPT5 * X ——> X, EXPF (X; EXP1),
^
        (Z[I] - MX[I] / SIGMA[I] ——> Y * Y ——> Y,
        NEGPT5 * Y ——> Y, EXPF (Y; EXP2),
^
        MX[I] * AREA[I] + Z[I] * (1.0 - AREA[I])
        + SIGØ2PI * EXP1 - SIGØ2PI * EXP2 ——> SZ,
^
   W[I] * SZ + SUMWSZ ——> SUMWSZ,
    I+1 ——> II
   {PRINT < II |||| Z[I] U SZ >}, SKIP: },
^
  DISCARD TABLE:
     , {<>,,,,,},
     , {<< ITEM | DISCARD | TABLE >>,,}
     , {<< |||| I U ||| W
          UU C U||| M U SIGMA
            >>,},
^
   FØR I = 0(1)NG-1
       {ITEM DISCARD[I] ——> J,
        ITEM DISCARD[I] + 1 ——> ITEM DISCARD[I],
        {PRINT < | ITEM DISCARD[I] ||| W[J] |||
        C[J] ||| MX[J] ||| SIGMA[J] >}, },
```

```
^
  FINISH : ..
^
9
  5
  JJ = 00

  ; PRINT PARAM : {
      , { << ||| I U W U C U M ||| SIGMA >>, },
    FØR J = 0(1) NN - 1
          { J+1 → JJ,
    { PRINT < || JJ ||| W[J] || C [J] ||| MX [J] ||| SIGMA [J]
        > }, }, } ..
^
9
  5
  A| = 0 *0 ,  B| = 0 * 0
  D| ELTATK = 0*0,
  N| UM = 0*0,  D| ENØM = 0*0,
  S| RT2 = 1.41421356 * 0,
  R| ØØT2PI = 0.39894228 *0,
  E| RF ;

  NØRMEX (AREA ; T) : {
          IF AREA = 0 : 0 → T, EXITNX . ;
          IF 0 > AREA U AREA ≥ 0.5 : ERRØR . ;
          1.0 → T,
  C|ØMPUTE :   T / SRT2 → A,  ERRØRF (A; ERF),
               0.5 * ERF - AREA → NUM,
                 A * A → A,  0 - A → A,
                 EXPF (A; A),
                 RØØT2PI * A → DENØM,
               NUM / DENØM → DELTATK,
               T - DELTATK → T,
               IF  T < 0 :  0 - T → T ;;

   DELTATK / T → B,
          0-0.00005 < B < 0.00005 :
                            EXITNX. CØMPUTE.
   E|RRØR : , { << ILLEGAL|VALUE|
                  ØF | AREA | IN |
                     NØRMEX | FUNCTIØN >> },
^
  E| XITNX : } ..
^
9
  5;
  ERRØRF (W. ERF. P.):
       { (((( 0.00328975 * W - 0.00039446)
        * W + 0.02743349 ) * W + 0.08864027)
        * W + 0.14112821) * W + 1.0 → P,
        P * P → P,   P * P → P,
        P * P → P,
        1.0 - 1.0 / P → ERF, } ..
^
9
```